Quivers. - General theory of A_∞-structure on quivers.

- Classification of twisted complexes for some specific examples.

Linear quiver L_n, Kronecker quiver K_2, Cyclic quiver C_n.

(Representation of these complexes as intersection theory)

A_∞-category $\mathcal{A}$. products $\mu : Hom(a_n, a_{n+1}) \otimes \cdots \otimes Hom(a_1, a_2) \to Hom(a_1, a_{n+1})$

$\mu_1 = d_{Hom(a,b)}$

μ_2 = Composition.

Ring R. ($\mathbb{K}$-algebra) consider $\underline{\text{Mod-}R}$.

→ $Res\, R = \{ M :$ right R-module $\}$. (choose P_M projective resolution for M)

$Hom^i_{Res R}(M, N) = \{$ chain maps $f : P_M \to P_N[i] \}$.

Mod-R = $H\, Res\, R$.

$\underset{\text{finitely generated}}{Mod_{fg}}\text{-}R \subseteq Mod\text{-}R$. $\underset{\text{finite dim. as } \mathbb{K}\text{-vector space}}{Mod_{fd}}\text{-}R \subseteq Mod\text{-}R$ full subcategory.

Twisted complexes. $\underline{\oplus_i \overset{\mathcal{A}}{M_i}[\overset{\mathbb{Z}}{k_i}],\ \delta \in Hom^1(\oplus_i M_i[k_i], \oplus_i M_i[k_i])}$ ← δ as matrix is lower triangular

$\underline{Hom}^d(\oplus_i M_i[k_i], \oplus_j N_j[l_j]) = \bigoplus_{i,j} Hom^{l_j - k_i + d}(M_i, N_j)$

satisfy MC-equality $\mu(\delta) + \mu(\delta, \delta) + \mu(\delta, \delta, \delta) + \cdots = 0$.

$\mu_{Tw}(\underset{Hom((M_n,\delta_n),(M_{n+1},\delta_{n+1}))}{f_n}, f_{n-1}, \cdots f_1) = \sum \mu(\underline{\delta_{n+1}, \cdots} f_n, \underline{\delta_n, \cdots} f_{n-1}, \cdots \cdots f_1, \delta_1, \cdots)$

can be written as chain complexes. e.g.

$\underline{M_1 \to M_2 \to M_3 \to \cdots}$

⚠ not as graded structure!

$M_1 \oplus M_2 \oplus M_3$

$\underline{Tw\mathcal{A}}$:= $\{ \oplus_i M_i[k_i] \}$ with higher products μ_{Tw}.

$D\mathcal{A} = H\, Tw\, \mathcal{A}$.

$D^0\mathcal{A} = H^0\, Tw\, \mathcal{A}$. ($D^0\, Mod\text{-}R = D^b\, \underset{\text{ordinary cat}}{\underline{Mod\text{-}R}}$)

$Mod\text{-}R \rightsquigarrow Ch^b Mod\text{-}R \overset{\text{homotopy category}}{\rightsquigarrow} K^b Mod\text{-}R \overset{\text{add morphisms}}{\rightsquigarrow} D^b Mod\text{-}R$

A_∞

Mod-R $\rightsquigarrow$ TwMod-R $\rightsquigarrow$ DMod-R

add objects. homotopy category

Quivers = Oriented graph.

Let Q be a quiver. then Q_0 denote the set of vertices.

Q_1 denote the set of edges. (arrows)

$s, t : Q_1 \to Q_0$.

source target

i.e. $a \in Q_1$. $a : \underset{Q_0}{s(a)} \to \underset{Q_0}{t(a)}$

path. p is a nontrivial path if p is a sequence of arrows. $a_1 \cdots a_k$

$\forall 1 \le i < k$, $t(a_i) = s(a_{i+1})$

p is a trivial path if p is a vertex. $\to s(v) = t(v) = v$

p is a cyclic path if p is non-trivial and $s(a_1) = t(a_k)$

path algebra. $\mathbb{C}Q = \mathbb{C}^{\oplus \text{all path}}$

multiplication $q \cdot p = \begin{cases} \circ \xrightarrow{p} \circ \xrightarrow{q} \circ & \text{if } s(q) = t(p) \\ 0 & \text{otherwise.} \end{cases}$

$Q_0 \subseteq \mathbb{C}Q$ set of orthogonal idempotents.

$v \cdot v = v$, $v \cdot w = 0$

also. $\sum_{v \in Q_0} v = id_{\mathbb{C}Q}$.

$\mathbb{C}Q_0 < \mathbb{C}Q$ semisimple subalgebra.

two-sided ideal $J = \langle a \mid a \in Q_1 \rangle \triangleleft \mathbb{C}Q$

path category $\mathbb{C}\mathcal{Q}$: $ob\, \mathbb{C}\mathcal{Q} = Q_0$.

$Hom(v, w) = \{\text{paths from } v \text{ to } w\} = w\mathbb{C}Qv$.

$id_v = v \in Hom(v, v)$.

Right $\mathbb{C}Q$-module $\sim$ right $\mathbb{C}\mathcal{Q}$-module

Modules of a $\mathbb{C}$-linear category $\mathcal{C}$.

Left modules are functors $\mathcal{C} \to Vect(\mathbb{C})$.

Right modules are functors $\mathcal{C}^{op} \to \mathrm{Vect}(\mathbb{C})$.

Let F is right $\mathbb{C}Q$-module, let $M = \bigoplus_{v \in Q_0} F(v)$

For a path p. $p: \begin{cases} F(t(p)) \xrightarrow{F(p)} F(s(p)) \\ F(v) \xrightarrow{0} 0 \quad (v \neq t(p)) \end{cases}$

Let M is right $\mathbb{C}Q$-module. consider $\mathcal{F}: \mathbb{C}Q^{op} \to \mathrm{Vect}(\mathbb{C})$

$$\mathcal{F}(v) = Mv$$

$$\mathcal{F}(p): F(t(p)) \to F(s(p))$$

$$mt(p) \mapsto mp = mt(p)p = mps(p).$$

$\mathrm{Mod}\, Q := \mathrm{Mod}\text{-}\mathbb{C}Q$. In $D\mathrm{Mod}\, Q$ (we want to analyze $\underline{D\mathrm{Mod}_{fg}(Q)}$)

quick example. $Q = C_1 = \circlearrowleft^{a}$ $a, aa, aaa, \ldots$

$\mathbb{C}Q \cong \mathbb{C}[x]$. $\mathrm{Spec}\, \mathbb{C}[x] = \mathbb{A}^1$

$D\mathrm{Coh}\, \mathbb{A}^1 \simeq D\mathrm{Mod}_{fg}\, \mathbb{C}[x] \simeq \underline{D\mathrm{Mod}_{fg}\, Q}$

- basic projective $\overset{v \in Q_0}{P_v} = \underset{\langle v \rangle}{\underline{v\mathbb{C}Q}} \in \mathrm{Mod}_{fg}\, Q$.

- basic simple $\overset{v \in Q_0}{S_v} = v\mathbb{C}Q/J = \mathbb{C}v \in \mathrm{Mod}_{fd}\, Q$.

Fact $\underline{\mathrm{Ext}^i_{\mathbb{C}Q}(P_v, P_w)} = \begin{cases} w\mathbb{C}Qv & \text{if } i = 0. \\ 0 & \text{otherwise.} \end{cases}$

$\Rightarrow$ consider $\underline{\mathcal{P}} \subseteq D\mathrm{Mod}\, Q$ generated by $\{P_v : v \in Q_0\}$.

then $\mathcal{P} \cong \mathbb{C}Q$ as A_∞-category.

Fact. S_v has projective resolution $\bigoplus_{s(a)=v} P_{t(a)} \xrightarrow{a} P_v$.

$$\mathrm{Ext}^i_{\mathbb{C}Q}(S_v, S_w) = \begin{cases} \delta_{vw}\mathbb{C} & \text{if } i = 0 \\ \mathbb{C}^{\oplus \#\{a : w \to v\}} & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases}$$

Let $S = \bigoplus_{v \in Q_0} S_v$ then $\mathrm{Ext}^\bullet_{\mathbb{C}Q}(S, S) = \dfrac{\mathbb{C}Q^{op}}{\langle a^* b^* \mid a^* b^* \in \mathbb{C}Q_1^{op} \rangle}$ $= \Lambda Q$ dual algebra (v^* has degree 0, a^* has degree 1)

$Q^{op} = \langle v^* : v \in Q_0,\ a^* : t(a) \to s(a) : a \in Q_1 \rangle$.

dual cat. $\Lambda\mathcal{Q}$ $a^*b^*=0$.

$\mathcal{S} \subseteq D\mathrm{Mod}\,Q$ generated by $\{S_v : v \in Q_0\}$.

then $\mathcal{S} \cong \Lambda\mathcal{Q}$ as A_∞ category.

Theorem: Let Q be a quiver. then we have

$$\underset{\underset{\text{trivial}}{\uparrow}}{D\mathcal{S} \subseteq} D\mathrm{Mod}_{fd}\,Q \subseteq \underset{\underset{\text{trivial.}}{\uparrow}}{D\mathcal{P} \subseteq} D\mathrm{Mod}_{fg}\,Q$$

all four are the same if $\underbrace{\overset{(\text{finite})}{Q} \text{ has no cycles.}}_{= \mathbb{C}Q \text{ finite dimensional}}$

proof. firstly show $\mathrm{Mod}_{fd}\,Q \subseteq \underline{D\mathcal{P}}$.

Let $M \in \mathrm{Mod}_{fd}\,Q$. consider

$$\bigoplus_{a \in Q_1} \underline{M_{s(a)} \otimes_{\mathbb{C}} P_{t(a)}} \xrightarrow{d} \bigoplus_{v \in Q_0} \underline{M_v \otimes_{\mathbb{C}} P_v}.$$

$$d\left(\textstyle\sum_a m_a \otimes x_a\right) = \textstyle\sum_a \left(m_a a \otimes x_a - m_a \otimes a x_a\right).$$

- projective resolution of $\underline{M}$.

- twisted complex in $D\mathcal{P}$.

secondly. if Q has no cycles. $\mathrm{Mod}_{fg}\,Q = \mathrm{Mod}_{fd}\,Q$.

- S_v are the only simple $\mathbb{C}Q$-modules.

$\Rightarrow$ P_v f.d., generated by S_v. $\Rightarrow$ $\mathcal{P} \subseteq D\mathcal{S}$.

Quiver representations. dimension vector $i : Q_0 \to \mathbb{N}$,

$$\underset{\text{right}}{\rho} \in \mathrm{Rep}(Q, i) = \bigoplus_{a \in Q_1} M_{i_{s(a)} \times i_{t(a)}}(\mathbb{C}).$$

as $\overset{\text{right}}{\mathbb{C}\mathcal{Q}}$-module $M_\rho : \mathbb{C}\mathcal{Q}^{op} \to \mathrm{Vect}(\mathbb{C})$.

$$M_\rho(v) = \mathbb{C}^{\oplus i(v)} \qquad M_\rho(a) = \mathbb{C}^{\oplus i_{t(a)}} \xrightarrow{\cdot \rho(a)} \mathbb{C}^{\oplus i_{s(a)}}$$

also as projective resolution over $\mathcal{P}$:

$$P_\rho = \bigoplus_{a \in Q_1} P_{t(a)}^{\oplus i_{s(a)}} \xrightarrow{\rho(a) - a} \bigoplus_{v \in Q_0} P_v^{\oplus i_v}$$

$\to$ only when M_ρ is a nilpotent module. as twisted complex $\in D\mathcal{S}$.

$$S_\rho = \Big(\bigoplus \underset{\underset{\mathbb{C}}{\cong}}{S_v}^{\oplus i_v},\ \delta = \sum_{a \in Q_1} \underbrace{\underset{i_{t(a)} \times i_{s(a)}}{\underset{\downarrow}{\rho(a)^T}}\ \underset{t(a) \to s(a)}{\underset{\downarrow}{a^*}}}_{\mathbb{C}^{t(a)} \to \mathbb{C}^{s(a)}}\Big). \Leftarrow$$

By explicit computation, in $\mathcal{DP} = P_\rho \simeq \mathcal{M}_\rho \cong \underbrace{S_\rho}_{\text{if exists.}}$

Theorem. every object in $\mathcal{DS}$ is isomorphic to a direct sum of shifts of S_ρ.

Generalisation: every object in $\mathrm{DMod}\,Q$ is isomorphic to a direct sum of shifts of $\mathbb{C}Q$-modules.

(from fact: $\mathbb{C}Q$ is hereditary algebra = submodules of projective $\mathbb{C}Q$-modules are also projective)

Example 1. Linear quiver L_n.

$$v_n \xleftarrow{a_{n-1}} v_{n-1} \leftarrow \cdots \xleftarrow{a_1} v_1$$

$$\Rightarrow \mathcal{S}: \quad S_n \xrightarrow{a_{n-1}^{*(1)}} S_{n-1} \xrightarrow{a_{n-2}^{*(1)}} \cdots \xrightarrow{a_1^{*(1)}} S_1$$

$id = v_n^*$, $id = v_{n-1}^*$, $id = v_1^*$

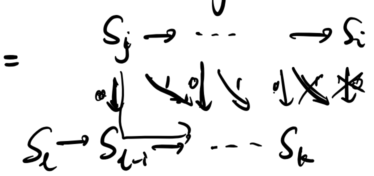
$\mathcal{DS} \ni S_{ij} = (S_i \oplus \cdots \oplus S_j,\ \delta = a_i^* + \cdots + a_{j-1}^*)$, $i \le j$

$$S_j \xrightarrow{a_{j-1}^*} S_{j-1} \to \cdots \xrightarrow{a_i^*} S_i$$

$\underline{H}\,\mathrm{Tw}\,\mathrm{Mod}\,Q$ $=$ $\mathcal{S} \subseteq \mathrm{DMod}\,Q$

Classification theorem for L_n: every indecomposable object in $\mathcal{DS}$ is isomorphic to shifts of some S_{ij}.

idea. $S_\rho = (S_i^{\oplus m_i} \oplus \cdots \oplus S_j^{\oplus m_j},\ \delta = \underline{B_i} a_i^* + \cdots + B_{j-1} a_{j-1}^*)$

$\underline{\mathrm{Hom}(S_{ij}, S_{kl})} =$  $= \overbrace{\bigoplus_{\substack{i\le u\le j\\ k\le u\le l}} \mathbb{C}\, id_{S_u}}^{0} \oplus \overbrace{\bigoplus_{\substack{i\le v+1\le j\\ k\le v\le l}} \mathbb{C}\, a_v^*}^{1}$

μ_{Tw}.

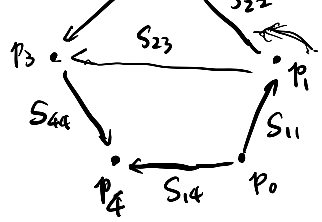
$\mu_{Tw}(a_v^*) = 0$. $\quad \underline{\mu_{Tw}(id_{S_u})} = \underset{0}{\mu(id_{S_u})} + \underline{\mu(a_{u-1}^*, id_{S_u})} - \underline{\mu(id_{S_u}, a_u^*)} + \overset{0}{\cdots}$

$= a_{u-1}^* - a_u^*$
(if exists) (if exists)

	$D^0\mathrm{Hom}(S_{ij}, S_{kl})$	$D^1\mathrm{Hom}(S_{ij}, S_{kl})$	$\cdots$
if $i+1 \le k-1 < j \le l$	$\mathbb{C}$	0	0
if $k-1 < i-1 \le l < j$	0	$\mathbb{C}$	0.
otherwise	0	0	0

$\mathcal{DS} \simeq \mathrm{DMod}_{fd}Q \cong \mathcal{DP} \simeq \mathrm{DMod}_{fg}Q$.

Intersection theory: consider $(n+1)$-gon with all diagonals.



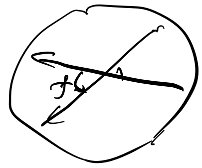
Let L_1, L_2 be two segments as above, with angle θ_1, θ_2 ($\in \mathbb{R}$) from the horizontal line

if L_1, L_2 intersect at point $p^\vee$ (inside polygon) with positive angle $\beta \in [0, \pi)$.

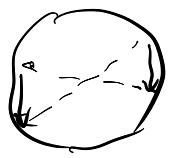
then $\mathrm{Hom}_{DJ}(L_1, L_2) = \mathbb{C}$ at degree $\dfrac{\beta - (\theta_1 - \theta_2)}{\pi}$

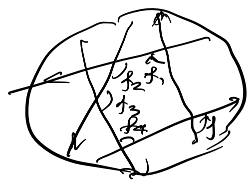
otherwise $\mathrm{Hom}_{DJ}(L_1, L_2) = 0$.

$\Rightarrow L_1[1] := L_1$ rotate by 2π



$\Rightarrow$ cone(f) =





$\Rightarrow \mu_{Tw}(f_4, f_3, f_2, f_1) = f$.

Kronecker Quiver. K_2.

$$\begin{tikzcd} v_1 & v_0 \arrow[l, bend right, "a_0"'] \arrow[l, bend left, "a_1"] \end{tikzcd}$$

$$\Rightarrow \mathcal{G} \quad \begin{tikzcd} S_1 \arrow[r, bend left, "a_0^*"] \arrow[r, bend right, "a_1^*"'] & S_0 \end{tikzcd}$$

DJ Classification theorem:

strings $\to S(i, w)$, $i \in \{0,1\}$, $w \in \mathbb{N}$: $\leftarrow \underbrace{S_1 \xrightarrow{a_0} S_0 \xleftarrow{a_1} S_1}_{} \xrightarrow{a_0} S_0 \xleftarrow{a_1} \cdots$

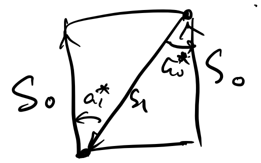
start at i, length is $w = 3$

bands $\to B(\lambda, n)$: $\left(S_1^{\oplus n} \oplus S_0^{\oplus n},\ \delta = 1_n a_0^* + J(\lambda, n) a_1^*\right)$ $\quad \lambda \neq 0$

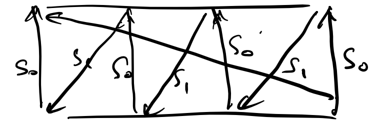
$$J(\lambda,n) = \begin{pmatrix} \lambda & & 0 \\ 1 & \lambda & \\ & \ddots & \ddots \\ 0 & & 1 \quad \lambda \end{pmatrix}$$

$\lim_{\lambda \to 0} B(\lambda, n) \cong S(0, 2n)$, $\lim_{\lambda \to \infty} B(\lambda, n) = S(1, 2n)$

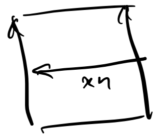
Intersection theory



$S(0,7)$



$B(\lambda, n)$



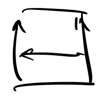
Local system: Let M be a manifold. a Local system is a rep. of its fundamental groupoid. $\mathcal{L}: \underbrace{\Pi_1(M)} \longrightarrow \mathrm{Vect}(\mathbb{C})$

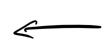
$\mathrm{Hom}(x, y) = \{\text{path } x \to y\}/\text{homotopy}$

$\Rightarrow$ Indecomposable Local system on S^1. $\xleftrightarrow{1:1}$ invertible Jordan blocks.

$\Pi_1(S^1) \simeq */\mathbb{Z}$. $\quad \mathcal{L}: * \longmapsto \mathbb{C}^{\oplus n}$

$1 \longmapsto J$

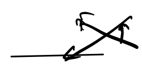
$\Rightarrow$ Let $B(\lambda, n)$ correspond to a local system 

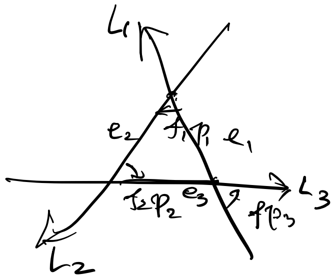
 is a trivial Local system $\mathcal{L}: * \longmapsto \mathbb{C}$

$S(\tau, \omega)$ $\quad \mathrm{id} \longmapsto \mathrm{id}$.

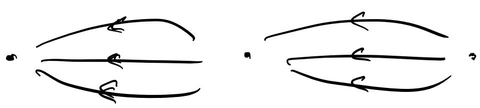
Local systems on cylinder

$\mathrm{Hom}_{DF}(L_1, L_2) = \bigoplus_{p \in L_1 \cap L_2} \mathrm{Hom}_{\mathbb{C}}(L_1(p), L_2(p))\, p.$

 $\Rightarrow$ degree.



$f = L_1(e_3) \circ f_2 \circ L_2(e_2) \circ f_1 \circ L_1(e_1)$



Df